

Thin Disk Theory with a Non-Zero Torque Boundary Condition and Comparisons with Simulations

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ABSTRACT

We present an analytical solution for thin disk accretion onto a Kerr black hole that extends the standard Novikov-Thorne α -disk in three ways: (i) it incorporates nonzero stresses at the inner edge of the disk, (ii) it extends into the plunging region, and (iii) it uses a corrected vertical gravity formula. The free parameters of the model are unchanged. Nonzero boundary stresses are included by replacing the Novikov-Thorne no torque boundary condition with the less strict requirement that the fluid velocity at the innermost stable circular orbit is the sound speed, which numerical models show to be the correct behavior for luminosities below $\sim 30\%$ Eddington. We assume the disk is thin so we can ignore advection. Boundary stresses scale as αh and advection terms scale as h^2 (where h is the disk opening angle ($h = H/r$)), so the model is self-consistent when $h < \alpha$. We compare our solution with slim disk models and general relativistic magnetohydrodynamic disk simulations. The model may improve the accuracy of black hole spin measurements.

Key words: accretion, accretion discs, black hole physics, hydrodynamics, (magnetohydrodynamics) MHD, gravitation

1 INTRODUCTION

The standard model for relativistic, thin disk accretion onto a black hole is the Novikov-Thorne (NT) model (Novikov & Thorne 1973; Page & Thorne 1974). It is the relativistic generalization of the Shakura & Sunyaev (1973) α -disk. The black hole is described by the Kerr metric with fixed mass, M , and specific angular momentum, a . The accretion flow is razor thin and confined to the equatorial plane so heat advection is negligible and vertical and radial energy transport can be decoupled. The disk has an inner edge at the innermost stable circular orbit (ISCO) where viscous stresses are assumed to vanish. Mass is accreted at a rate \dot{M} as stresses and radiation transport energy and angular momentum outwards. The NT disk has four free parameters: M , a , \dot{M} , and α .

Slim disk models generalize the NT model to include heat advection and coupled radial and vertical energy transport. They are solved numerically by requiring the flow to pass smoothly through a sonic point (Abramowicz et al. 1988; Paczyński 2000; Afshordi & Paczyński 2003; Shafee et al. 2008b; Sądowski 2009; Abramowicz et al. 2010; Sądowski et al. 2011). They have the same free parameters as the NT disk.

General relativistic, magnetohydrodynamic (GRMHD) thin disk simulations (Shafee et al. 2008a; Noble & Krolik 2009; Noble et al. 2010; Penna et al. 2010) incorporate magnetic fields and turbulence is driven by the magnetorotational instability (MRI) (Balbus & Hawley 1991, 1998). Radiation physics is described in an ad-hoc way as a sink term in the energy equation that tends to drive the entropy of the fluid towards a target, K_c . The MRI generates turbulence self-consistently, so the α -viscosity prescription is not used. However the strength of the saturated magnetic stresses, the “effective α ,” depends on the gas-to-magnetic pressure ratio, β_i , of the initial fields in the simulation. So the target entropy, K_c , plays a similar role to \dot{M} , and β_i is analogous to α .

Noble et al. (2010) presented simulations of GRMHD disks with different thicknesses and compared their stress profiles. They argued there is a large stress at the inner edge even in the limit of vanishing disk opening angle $h \rightarrow 0$. They concluded magnetized disks cannot be described by the NT model independently of disk thickness. Penna et al. (2010) performed similar simulations but found the stress at the inner edge to be directly proportional to thickness. They argued the NT zero-stress boundary condition is valid in the limit $h \rightarrow 0$. We believe the difference is in their definitions of the stress. Noble et al. (2010) included all of the fluid. Penna et al. (2010) made a distinction between the high density disk region and the low density, highly magnetized, coronal region.

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They included only disk fluid in stress calculations. The corona, if it is included, makes a large contribution to the stress.

The present model only describes the high-density disk region, which is expected to dominate the emission leading to the observed thermal spectral component. The corona is expected to contribute mostly to non-thermal spectral components. So the Penna et al. (2010) result that stress scales with h is the relevant one.

Our model includes a nonzero stress at the inner edge of the disk. Nonzero stress boundary conditions have been previously considered in the context of Newtonian (Shapiro & Teukolsky 1983) and general relativistic (Agol & Krolik 2000) accretion. However, the stress at the inner edge is a free parameter in these models. We eliminate this parameter by identifying the inner edge with the sonic point and relating the stress there to the sound speed. This prescription reduces to the NT zero-torque boundary condition in the razor thin limit $h \rightarrow 0$.

In the next section we describe the differences between our model and NT. In §3 we give the explicit disk solution. In §4 we compare it to slim disk models and in §5 we compare it to GRMHD disk simulations. We summarize our main results in §6. The Kerr metric and the disk structure equations are summarized in the appendices. A Fortran code which computes our thin disk solutions is available at <https://www.cfa.harvard.edu/~rpenna/thindisk>.

2 PHYSICS BEYOND THE STANDARD DISK MODEL

Our model extends the NT model in three ways: (i) it incorporates nonzero stresses at the inner edge of the disk, (ii) it extends into the plunging region, and (iii) it uses the correct vertical gravity formula. In this section we discuss each of these developments.

2.1 The inner edge boundary condition

The criterion $h \ll \alpha$, where h is the disk opening angle ($h = H/r$) and α is the “effective viscosity” parameter (c.f. §5), governs the structure of weakly-magnetized GRMHD disk simulations. When the disk is thin, $h \ll \alpha$, the surface density has an inner edge near the sonic point. When the disk is thick, $h \gg \alpha$, advection causes the disk density to increase monotonically down to the event horizon. This is illustrated in Figures 1 and 2, which show time-averaged rest mass density in the $r - z$ plane for eight GRMHD simulations. Rest mass density drops as the disk approaches the sonic point if and only if $h \ll \alpha$.

Thick disks are non-Keplerian and insensitive to the ISCO. The sonic point and ISCO only coincide if the disk is thin. Abramowicz et al. (2010) find the Boyer-Lindquist radial positions of the sonic point and ISCO deviate by $< 3\%$, independently of α , for $a_* = 0$ slim disks with $\dot{M}/\dot{M}_{\text{edd}} < 0.3$.

Summarizing these two observations: the inner edge, sonic point, and ISCO are at the same radius if $h \ll \alpha$ and $\dot{M}/\dot{M}_{\text{edd}} < 0.3$. We will assume these conditions hold for thin disks. Under these conditions, energy advection and energy generation by compression, which scale as h^2 , can be neglected relative to boundary stresses at the ISCO, which scale as ah . We give a proof in Appendix C. The NT model ignores boundary stresses at the ISCO, which is valid in the limit $h \rightarrow 0$. By including them, we obtain disk solutions with nonzero stress and flux at the ISCO, and radial velocity and surface density profiles that can be extended into the plunging region.

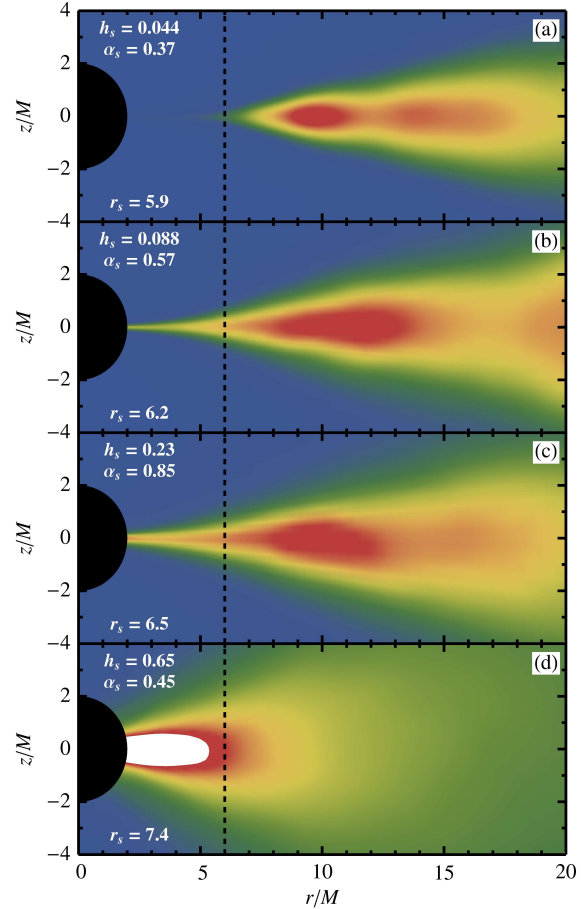


Figure 1. Time-averaged rest mass density in the $r - z$ plane for four GRMHD simulations with $a_* = 0$ and various disk thicknesses. The dashed vertical line marks the ISCO. The disk opening angle, h , and effective Shakura-Sunyaev viscosity, α , are measured at the sonic point, r_0 (c.f. §5). The top three panels have $h \ll \alpha$ and the inner edge of the disk is located outside the ISCO. The lowermost panel has $h \gg \alpha$ and the density increases monotonically down to the event horizon. Panels (a) and (b) are the thin and thick $a_* = 0$ simulations of Kulkarni et al. (2011). Panels (c) and (d) are models A0HR2 and A0HR3, respectively, from Penna et al. (2010).

2.2 The plunging region

We assume magnetic fields are weak and plunging region stresses are small, so the fluid motion inside the ISCO can be approximated by geodesics. Krolik (1999) argued magnetic fields are always dynamically important inside the ISCO and geodesic trajectories are never a good approximation there. However we will show in §5 that our solution gives a good fit to the radial velocity profile of the fiducial $a_* = 0$ GRMHD simulation of Kulkarni et al. (2011). So GRMHD disks do exist in which the field is sufficiently weak inside the ISCO that geodesic motion is a good approximation.

To solve the geodesic equations (A5a)-(A5c) for the fluid motion, we need to fix the fluid energy $E = |u_t|$ and angular momentum $L = u_\phi$. We assume the fluid plunges with constant E and L , so it suffices to fix the fluid energy and angular momentum at the ISCO. Equivalently, it is enough to specify the fluid velocity at the ISCO.

We assume the angular velocity at the ISCO is Keplerian (A8)

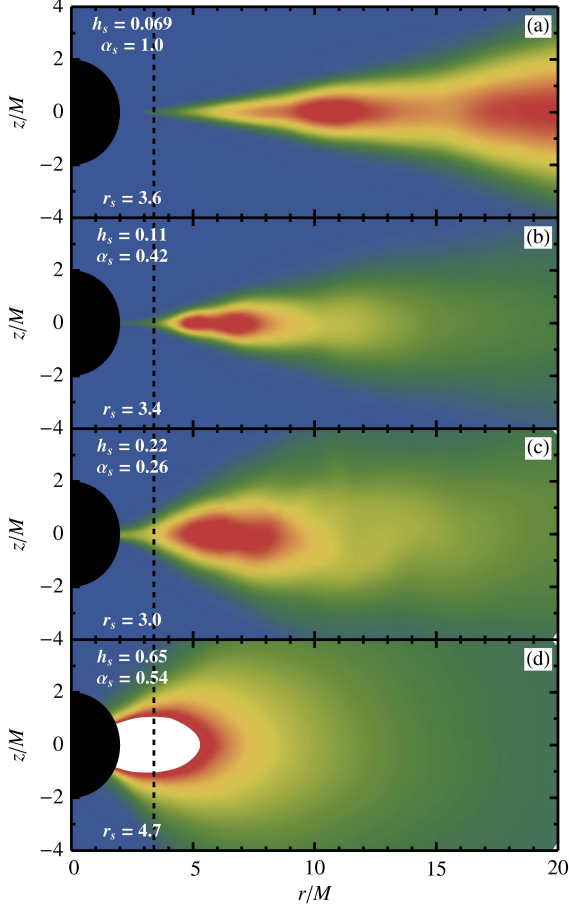


Figure 2. Same as Figure 1 but with $a_* = 0.7$. Panel (a) is the $a_* = 0.7$ simulation of Kulkarni et al. (2011). Panels (b), (c), and (d) are models A7HR1, A7HR2, and A7HR3, respectively, from Penna et al. (2010).

and the disk is in the equatorial plane, so $u^\theta = 0$. The radial velocity at the ISCO is the sound speed (c.f. §2.1):

$$V_0 = \left(h \frac{\mathcal{L}_*}{r} \right)_0. \quad (2.1)$$

This is the radial velocity in the fluid frame. It is related to u^r by $V = \sqrt{g_{rr}} u^r / \left(1 + \left(\sqrt{g_{rr}} u^r \right)^2 \right)^{1/2}$.

The flow inside the ISCO is now fixed by the disk structure equations. See Appendix B for details.

2.3 Vertical gravity

NT gave an incorrect formula for the “vertical gravity” appearing in the pressure balance equation, creating errors in the disk solution. Better formulae for the vertical gravity were found by Eardley & Lightman (1975) and Riffert & Herold (1995). However

they assume the disk follows circular geodesics, so their results break down in the plunging region. Abramowicz et al. (1997) found a more general formula which is valid in the plunging region. We use this result in the disk structure equations (B12).

3 EXPLICIT DISK SOLUTIONS

The model has four free parameters:

- M = mass of black hole,
- a = specific angular momentum of hole,
- \dot{M} = accretion rate,
- α = Shakura-Sunyaev viscosity.

These are taken to be constants. Following NT, we shall express M in units of $3M_\odot$ and we shall express \dot{M} in units of 10^{17} g/sec:

$$M_* \equiv M/3M_\odot, \quad \dot{M}_* \equiv \dot{M}/10^{17} \text{ g sec}^{-1}.$$

The metric and disk structure equations are summarized in Appendices A and B. In this section we give the solutions for the quantities that appear in the structure equations¹: flux of radiant energy off the upper surface of the disk, F , surface density, Σ , disk thickness, H , rest mass density in the local rest frame, ρ , temperature, T , and radial velocity in the locally nonrotating frame, $v^{\hat{r}}$. In addition to quantities that appear explicitly in the equations of structure, we calculate the optical depth at the center of the disk,

$$\tau = \bar{\kappa} \Sigma, \quad (3.1)$$

and the characteristic timescale for the gas to move inward from radius r to the inner edge of the disk,

$$\Delta t(r) = -r/v^{\hat{r}}. \quad (3.2)$$

The disk outside the ISCO can be divided into 4 regions: an “outer region” (large radii) in which gas pressure dominates over radiation pressure, and in which the opacity is predominantly free-free; a “middle region” (smaller radii) in which gas pressure dominates over radiation pressure, but opacity is predominantly due to electron scattering; an “inner region” (even smaller radii) in which radiation pressure dominates over gas pressure, and opacity is predominantly due to electron scattering; and an “edge region” (smallest radii) where gas pressure again dominates over radiation pressure, and opacity is predominantly due to electron scattering.

The NT model does not include the edge region explicitly. However it must exist because the no-torque boundary condition implies radiant flux and radiation pressure go to zero at the ISCO. The NT inner region surface density is singular at the ISCO because of this inconsistency. The edge region surface density is finite in the NT limit.

The solutions are functions of the dimensionless radial coordinate $x = \sqrt{r/M}$. Calligraphic letters denote functions of x and a with value unity far from the hole. A subscript 0 indicates the quantity is evaluated at the ISCO.

¹ A Fortran code which computes our thin disk solutions is available at <https://www.cfa.harvard.edu/~rpenna/thindisk>.

3.1 Plunging region

$p = p^{(\text{gas})}$, $\bar{\kappa} = \bar{\kappa}_{es}$. In this region the equations of structure (B9)-(B16) yield the following radial profiles:

$$F = (2 \times 10^{18} \text{ erg/cm}^2 \text{ sec}) (\alpha^{4/3} M_*^{-3} \dot{M}_* 5/3) x^{-26/3} x_0^{4/3} \mathcal{D}^{-5/6} \mathcal{K}^{4/3} \mathcal{F}_0^{4/3} \mathcal{G}_0^{-4/3} v_*^{-5/3}, \quad (3.3a)$$

$$\Sigma = (1 \text{ g/cm}^2) (M_*^{-1} \dot{M}_*) x^{-2} \mathcal{D}^{-1/2} v_*^{-1}, \quad (3.3b)$$

$$H = (60 \text{ cm}) (\alpha^{1/6} M_*^{1/2} \dot{M}_*^{1/3}) x^{8/3} x_0^{-5/6} \mathcal{D}^{-1/6} \mathcal{K}^{1/6} \mathcal{F}_0^{1/6} \mathcal{G}_0^{-1/6} \mathcal{R}_0^{-1/2} v_*^{-1/3} \quad (3.3c)$$

$$\rho = (0.01 \text{ g/cm}^3) (\alpha^{-1/6} M_*^{-3/2} \dot{M}_*^{2/3}) x^{-14/3} x_0^{5/6} \mathcal{D}^{-1/3} \mathcal{K}^{-1/6} \mathcal{F}_0^{-1/6} \mathcal{G}_0^{1/6} \mathcal{R}_0^{1/2} v_*^{-2/3} \quad (3.3d)$$

$$T = (2 \times 10^5 \text{ K}) (\alpha^{1/3} M_*^{-1} \dot{M}_*^{2/3}) x^{-8/3} x_0^{1/3} \mathcal{D}^{-1/3} \mathcal{K}^{1/3} \mathcal{F}_0^{1/3} \mathcal{G}_0^{-1/3} v_*^{-2/3} \quad (3.3e)$$

$$\tau_{es} = 0.3 (M_*^{-1} \dot{M}_*) x^{-2} \mathcal{D}^{-1/2} v_*^{-1}, \quad (3.3f)$$

$$\Delta t(r) = (1 \times 10^{-5} \text{ sec}) (M_*) x^2 v_*^{-1}, \quad (3.3g)$$

$$v^{\hat{r}} = -(3 \times 10^{10} \text{ cm/sec}) v_*. \quad (3.3h)$$

We have defined the dimensionless radial velocity profile:

$$v_* = \left[\left[\mathcal{C}_0^{-1} \mathcal{G}_0^2 \mathcal{V} - 1 \right] + (7 \times 10^{-3}) (\alpha^{1/4} M_*^{-3/4} \dot{M}_*^{1/2}) x_0^{-7/4} \mathcal{C}_0^{-5/4} \mathcal{D}_0^{-1} \mathcal{G}_0^2 \mathcal{V} \right]^{1/2} \quad (3.4)$$

The term in square brackets dominates near the horizon. The term proportional to \dot{M} dominates near the ISCO. At the horizon the radial velocity is c and at the ISCO it is the sound speed. In the limit $\dot{M}/\dot{M}_{\text{edd}} \rightarrow 0$, we may set $v_* = \left[\mathcal{C}_0^{-1} \mathcal{G}_0^2 \mathcal{V} - 1 \right]^{1/2}$ and the gas is released from rest at the ISCO.

3.2 Edge region

$p = p^{(\text{gas})}$, $\bar{\kappa} = \bar{\kappa}_{es}$. In this region the equations of structure (B9)-(B16) yield:

$$F = (0.6 \times 10^{26} \text{ erg/cm}^2 \text{ sec}) (M_*^{-2} \dot{M}_*) x^{-6} \mathcal{B}^{-1} \mathcal{C}^{-1/2} \Phi, \quad (3.5a)$$

$$\Sigma = (5 \times 10^4 \text{ g/cm}^2) (\alpha^{-4/5} M_*^{-2/5} \dot{M}_*^{3/5}) x^{-6/5} \mathcal{B}^{-4/5} \mathcal{C}^{-1/2} \mathcal{D}^{-4/5} \Phi^{3/5}, \quad (3.5b)$$

$$H = (3 \times 10^3 \text{ cm}) (\alpha^{-1/10} M_*^{7/10} \dot{M}_*^{1/5}) x^{21/10} \mathcal{A} \mathcal{B}^{-6/5} \mathcal{C}^{1/2} \mathcal{D}^{-3/5} \mathcal{S}^{-1/2} \Phi^{1/5}, \quad (3.5c)$$

$$\rho = (10 \text{ g/cm}^3) (\alpha^{-7/10} M_*^{-11/10} \dot{M}_*^{2/5}) x^{-33/10} \mathcal{A}^{-1} \mathcal{B}^{3/5} \mathcal{D}^{-1/5} \mathcal{S}^{1/2} \Phi^{2/5}, \quad (3.5d)$$

$$T = (3 \times 10^8 \text{ K}) (\alpha^{-1/5} M_*^{-3/5} \dot{M}_*^{2/5}) x^{-9/5} \mathcal{B}^{-2/5} \mathcal{D}^{-1/5} \Phi^{2/5}, \quad (3.5e)$$

$$\tau_{es} = (2 \times 10^4) (\alpha^{-4/5} M_*^{-2/5} \dot{M}_*^{3/5}) x^{-6/5} \mathcal{B}^{-3/5} \mathcal{C}^{1/2} \mathcal{D}^{-4/5} \Phi^{3/5}, \quad (3.5f)$$

$$\frac{\tau_{ff}}{\tau_{es}} = (0.6 \times 10^{-5}) (M_* \dot{M}_*^{-1}) x^3 \mathcal{A}^{-1} \mathcal{B}^2 \mathcal{D}^{1/2} \mathcal{S}^{1/2} \Phi^{-1}, \quad (3.5g)$$

$$\Delta t(r) = (0.7 \text{ sec}) (\alpha^{-4/5} M_*^{8/5} \dot{M}_*^{-2/5}) x^{14/5} \mathcal{B}^{-4/5} \mathcal{C}^{1/2} \mathcal{D}^{-3/10} \Phi^{3/5}, \quad (3.5h)$$

$$v^{\hat{r}} = -(6 \times 10^5 \text{ cm/sec}) (\alpha^{4/5} M_*^{-3/5} \dot{M}_*^{2/5}) x^{-4/5} \mathcal{B}^{4/5} \mathcal{C}^{-1/2} \mathcal{D}^{3/10} \Phi^{-3/5}. \quad (3.5i)$$

We have defined a new function:

$$\Phi = \mathcal{Q} + (0.02) (\alpha^{9/8} M_*^{-3/8} \dot{M}_*^{1/4}) x^{-1} \mathcal{B} \mathcal{C}^{-1/2} (x_0^{9/8} \mathcal{C}_0^{-5/8} \mathcal{G}_0 \mathcal{V}_0^{1/2}), \quad (3.6)$$

which controls the shape of the radiant flux profile, $F(r)$. At large distances, the first term on the RHS is order unity, and the second term decays as x^{-1} . Near the ISCO, the first term goes to zero, and the second term is nonzero and proportional to $\dot{M}^{1/4}$. So if $\dot{M}/\dot{M}_{\text{edd}}$ is small, then the second term is small everywhere and we may substitute $\Phi \rightarrow \mathcal{Q}$. This gives the NT flux. Quantities which depend on \mathcal{S} will still differ from NT because our vertical gravity prescription is different (c.f. §2.3). We may revert to the incorrect NT vertical gravity with the substitution $\mathcal{S} \rightarrow \mathcal{E}$. With these two substitutions our model becomes the NT solution outside the plunging region.

3.3 Inner region

$p = p^{(\text{rad})}$, $\bar{\kappa} = \bar{\kappa}_{es}$. In this region the equations of structure (B9)-(B16) yield:

$$F = (0.6 \times 10^{26} \text{ erg/cm}^2 \text{ sec}) (M_*^{-2} \dot{M}_*) x^{-6} \mathcal{B}^{-1} \mathcal{C}^{-1/2} \Phi, \quad (3.7a)$$

$$\Sigma = (20 \text{ g/cm}^2) (\alpha^{-1} M_* \dot{M}_*^{-1}) x^3 \mathcal{A}^{-2} \mathcal{B}^3 \mathcal{C}^{1/2} \mathcal{S} \Phi^{-1}, \quad (3.7b)$$

$$H = (1 \times 10^5 \text{ cm}) (\dot{M}_*) \mathcal{A}^2 \mathcal{B}^{-3} \mathcal{C}^{1/2} \mathcal{D}^{-1} \mathcal{S}^{-1} \Phi, \quad (3.7c)$$

$$\rho = (1 \times 10^{-4} \text{ g/cm}^3) (\alpha^{-1} M_* \dot{M}_*^{-2}) x^3 \mathcal{A}^{-4} \mathcal{B}^6 \mathcal{D} \mathcal{S}^2 \Phi^{-2}, \quad (3.7d)$$

$$T = (4 \times 10^7 \text{ K}) (\alpha^{-1/4} M_*^{-1/4}) x^{-3/4} \mathcal{A}^{-1/2} \mathcal{B}^{1/2} \mathcal{S}^{1/4}, \quad (3.7e)$$

$$\tau_{es} = 8 \left(\alpha^{-1} M_* \dot{M}_*^{-1} \right) x^3 \mathcal{A}^{-2} \mathcal{B}^3 \mathcal{C}^{1/2} \mathcal{S} \Phi^{-1}, \quad (3.7f)$$

$$\frac{p^{(\text{gas})}}{p^{(\text{rad})}} = \left(5 \times 10^{-5} \right) \left(\alpha^{-1/4} M_*^{7/4} \dot{M}_*^{-2} \right) x^{21/4} \mathcal{A}^{-5/2} \mathcal{B}^{9/2} \mathcal{D} \mathcal{S}^{5/4} \Phi^{-2}, \quad (3.7g)$$

$$\Delta t(r) = \left(2 \times 10^{-4} \text{ sec} \right) \left(\alpha^{-1} M_*^3 \dot{M}_*^{-2} \right) x^7 \mathcal{A}^{-2} \mathcal{B}^3 \mathcal{C}^{1/2} \mathcal{D}^{1/2} \mathcal{S} \Phi^{-1}, \quad (3.7h)$$

$$v^{\hat{r}} = - \left(2 \times 10^9 \text{ cm/sec} \right) \left(\alpha M_*^{-2} \dot{M}_*^2 \right) x^{-5} \mathcal{A}^2 \mathcal{B}^{-3} \mathcal{C}^{-1/2} \mathcal{D}^{-1/2} \mathcal{S}^{-1} \Phi. \quad (3.7i)$$

$$(3.7j)$$

The boundaries between the edge, inner, and middle regions can be computed from the ratio of pressures (3.7g).

3.4 Middle region

$p = p^{(\text{gas})}$, $\bar{\kappa} = \bar{\kappa}_{es}$. The solution is the same as the edge region solution (c.f. §3.2).

3.5 Outer region

$p = p^{(\text{gas})}$, $\bar{\kappa} = \bar{\kappa}_{ff}$. In this region the equations of structure (B9)-(B16) yield:

$$F = \left(0.6 \times 10^{26} \text{ erg/cm}^2 \text{ sec} \right) \left(M_*^{-2} \dot{M}_* \right) x^{-6} \mathcal{B}^{-1} \mathcal{C}^{-1/2} \Phi, \quad (3.8a)$$

$$\Sigma = \left(2 \times 10^5 \text{ g/cm}^2 \right) \left(\alpha^{-4/5} M_*^{-1/2} \dot{M}_*^{7/10} \right) x^{-3/2} \mathcal{A}^{1/10} \mathcal{B}^{-4/5} \mathcal{C}^{1/2} \mathcal{D}^{-17/20} \mathcal{S}^{-1/20} \Phi^{7/10}, \quad (3.8b)$$

$$H = \left(9 \times 10^2 \text{ cm} \right) \left(\alpha^{-1/10} M_*^{3/4} \dot{M}_*^{3/20} \right) x^{9/4} \mathcal{A}^{19/20} \mathcal{B}^{-11/10} \mathcal{C}^{1/2} \mathcal{D}^{-23/40} \mathcal{S}^{-19/40} \Phi^{3/20}, \quad (3.8c)$$

$$\rho = \left(80 \text{ g/cm}^3 \right) \left(\alpha^{-7/10} M_*^{-5/4} \dot{M}_*^{11/20} \right) x^{-15/4} \mathcal{A}^{-17/20} \mathcal{B}^{3/10} \mathcal{D}^{-11/40} \mathcal{S}^{17/40} \Phi^{11/20}, \quad (3.8d)$$

$$T = \left(8 \times 10^7 \text{ K} \right) \left(\alpha^{-1/5} M_*^{-1/2} \dot{M}_*^{3/10} \right) x^{-3/2} \mathcal{A}^{-1/10} \mathcal{B}^{-1/5} \mathcal{D}^{-3/20} \mathcal{S}^{1/20} \Phi^{3/10}, \quad (3.8e)$$

$$\tau_{ff} = \left(2 \times 10^2 \right) \left(\alpha^{-4/5} \dot{M}_*^{1/5} \right) \mathcal{A}^{-2/5} \mathcal{B}^{1/5} \mathcal{C}^{1/2} \mathcal{D}^{-3/5} \mathcal{S}^{1/5} \Phi^{1/5}, \quad (3.8f)$$

$$\frac{\tau_{ff}}{\tau_{es}} = 3 \times 10^{-3} \left(M_*^{1/2} \dot{M}_*^{-1/2} \right) x^{3/2} \mathcal{A}^{-1/2} \mathcal{B}^{2/5} \mathcal{D}^{1/4} \mathcal{S}^{1/4} \Phi^{-1/2}, \quad (3.8g)$$

$$\Delta t(r) = \left(2 \text{ sec} \right) \left(\alpha^{-4/5} M_*^{3/2} \dot{M}_*^{-3/10} \right) x^{5/2} \mathcal{A}^{1/10} \mathcal{B}^{-4/5} \mathcal{C}^{1/2} \mathcal{D}^{-7/20} \mathcal{S}^{-1/20} \Phi^{7/10}, \quad (3.8h)$$

$$v^{\hat{r}} = - \left(2 \times 10^5 \text{ cm/sec} \right) \left(\alpha^{4/5} M_*^{-1/2} \dot{M}_*^{3/10} \right) x^{-1/2} \mathcal{A}^{-1/10} \mathcal{B}^{4/5} \mathcal{C}^{-1/2} \mathcal{D}^{7/20} \mathcal{S}^{1/20} \Phi^{-7/10}. \quad (3.8i)$$

$$(3.8j)$$

4 COMPARISON WITH SLIM DISK MODELS

Slim disk solutions include advection. They are computed numerically subject to the condition that the flow pass smoothly through a sonic point. The position of the sonic point is free to vary. A model for energy transport in the vertical direction, including radiative transport and convection, is coupled to the radial equations. For a complete description see Sądowski et al. (2011).

When in the thin disk regime, $h \ll \alpha$ and $\dot{M}/\dot{M}_{\text{edd}} < 0.3$, the sonic point is near the ISCO and advection can be neglected (c.f. §2.1). This enables the analytical solution of §3. So we expect our model and slim disk models to be similar under these conditions. We make this comparison in Figures 3 and 4. The NT disk is also included.

The disk solutions in Figure 3 have $M = 10M_{\odot}$, $a_* = 0$, $\alpha = 0.1$ and $\dot{M}/\dot{M}_{\text{edd}} = 0.3$. Our model and slim disk solutions are in good agreement inside the ISCO where the NT model does not extend. The slim disk surface density is larger outside the ISCO because it includes both radiation and gas pressure (which are comparable there), while the analytical disk models include only radiation pressure at these radii. All three models eventually converge at large radii. Figure 4 is the same except with $a_* = 0.5$.

5 COMPARISON WITH GRMHD SIMULATIONS

In this section, we compare the fiducial, $a_* = 0$ GRMHD simulation of Kulkarni et al. (2011) with analytical thin disk solutions. The target entropy in the GRMHD cooling function was $K_c = 0.00034$ and the gas-to-magnetic pressure ratio of the initial fields in the simulation was $\beta_i = 100$. These parameters play similar roles to \dot{M} and α in the thin disk solutions. The simulation reached $t = 26300M$. At large radii, the viscous timescale is long and the solution is not converged. The estimates in Penna et al. (2010) suggest steady state is reached out to $r = 9M$.

5.1 Hydrostatic equilibrium of GRMHD disks

In hydrostatic equilibrium, opening angle is related to pressure and density by (B12):

$$h = \sqrt{\frac{p}{\rho}} \frac{r}{\mathcal{L}_*}, \quad (5.1)$$

where $\mathcal{L}_*^2 = u_{\phi}^2 - a^2 (u_t - 1)$ (Abramowicz et al. 1997). Our thin disk solutions assume hydrostatic equilibrium even inside the plunging region, so we first check whether this is a good description of GRMHD disks.

A popular definition of opening angle for GRMHD disks is

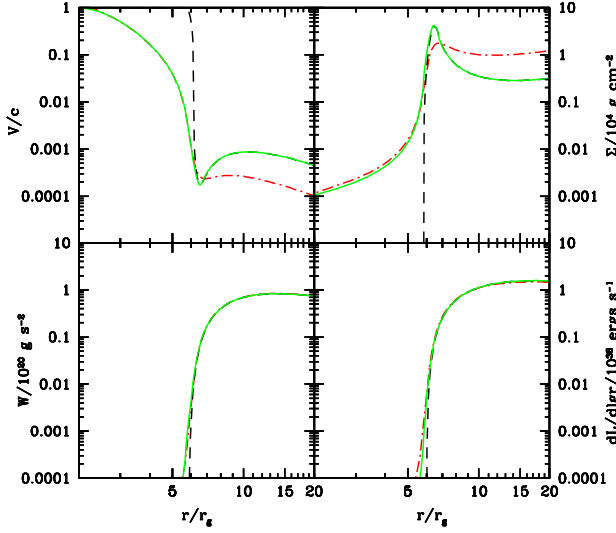


Figure 3. Radial velocity, surface density, stress, and luminosity versus radius for three disk models: NT (dashed black), slim disk (dot-dashed red), and our generalized thin disk (solid green). These solutions have $M = 10M_\odot$, $a_* = 0$, $\alpha = 0.1$, and $\dot{M}/\dot{M}_{\text{edd}} = 0.3$. The NT solution terminates at the ISCO but the slim disk and our model continue to the event horizon.

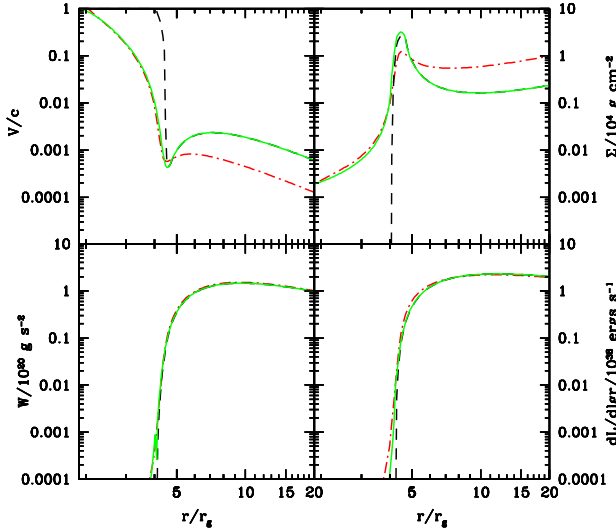


Figure 4. Same as Figure 3 but with $a_* = 0.5$.

(Penna et al. 2010; Noble et al. 2010; Shafee et al. 2008a):

$$h^{(\text{rms})} \equiv \left(\frac{\int (\theta - \pi/2)^2 \rho \sqrt{-g} dt d\theta d\phi}{\int \rho \sqrt{-g} dt d\theta d\phi} \right)^{1/2}. \quad (5.2)$$

In Figure 5, we compare $h^{(\text{rms})}$ for the fiducial GRMHD disk (dotted red) with the RHS of (5.1), the opening angle expected from hydrostatic equilibrium (dot-dashed blue). The latter is computed from the rest mass density and total pressure at the midplane. (Replacing midplane values with density-weighted vertical averages has little effect.) Hydrostatic equilibrium appears to be a bad approximation

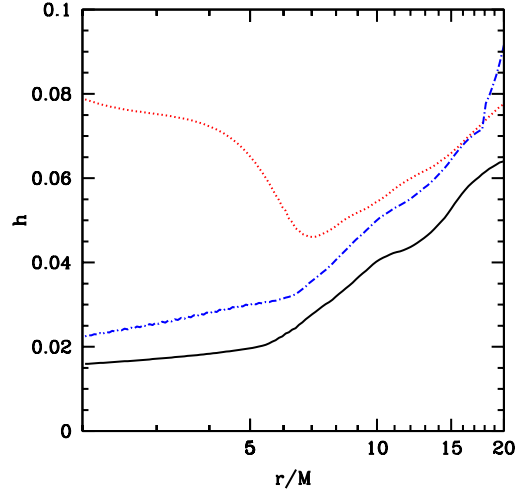


Figure 5. Unnormalized, ρ -weighted opening angle $h^{(\text{rms})}$ (dotted red), normalized, ρ^2 -weighted opening angle $h_2^{(\text{rms})}$ (solid black), and opening angle expected from hydrostatic equilibrium (dot-dashed blue). The opening angle expected from hydrostatic equilibrium is within 30% of $h_2^{(\text{rms})}$ at all converged radii.

in the plunging region, where it gives the wrong opening angle by as much as a factor of 4.

The simulation's high density, gas pressure dominated disk is surrounded by a low density, magnetically supported corona. These two regions have different scale heights and the apparent deviations from hydrostatic equilibrium could be a result of mixing them in the definition of opening angle. Our thin disk model only applies to the disk region, so we would like to minimize the contribution of the corona. We can do this by weighting the integrals in (5.2) with higher powers of ρ , because this concentrates attention on high density regions. Unfortunately, the opening angle then depends on this choice: higher powers of ρ give smaller opening angles. To get an invariant measure, we normalize the opening angle as follows. The vertical density profile of a polytropic gas in hydrostatic equilibrium is

$$\rho(z) = \rho(z=0) \left(1 - \frac{(z/r)^2}{\mathcal{H}^2} \right)^N, \quad (5.3)$$

where \mathcal{H} is the opening angle. The simulation has $\Gamma = 1 + 1/N = 4/3$, so $N = 3$. We normalize our definition of opening angle such that it returns \mathcal{H} when given the analytical solution (5.3). So the normalized, ρ^2 -weighted opening angle is

$$h_2^{(\text{rms})} \equiv \frac{\sqrt{15}}{3} \left(\frac{\int (\theta - \pi/2)^2 \rho^2 \sqrt{-g} dt d\theta d\phi}{\int \rho^2 \sqrt{-g} dt d\theta d\phi} \right)^{1/2}, \quad (5.4)$$

where $\sqrt{15}/3$ is the normalization defined by (5.3).

We plot $h_2^{(\text{rms})}$ for the fiducial GRMHD simulation in Figure 5 (solid black). It is within 30% of the opening angle expected from hydrostatic equilibrium (dot-dashed blue) at all converged radii. Emphasizing the disk over the corona has removed the discrepancy and shows that hydrostatic equilibrium is a good approximation in the disk region. From now on, we define the GRMHD opening angle to be $h = h_2^{(\text{rms})}$.

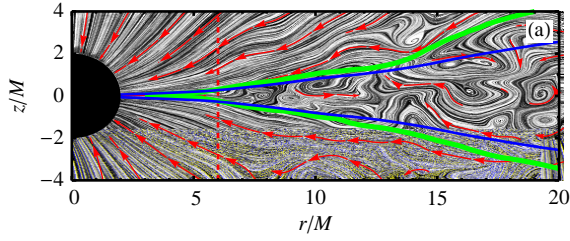


Figure 6. A snapshot of the velocity streamlines in the saturated state of the fiducial GRMHD disk simulation. The $\beta = 1$ contour (green) divides the high density, weakly magnetized disk from the low density, highly magnetized coronal and plunging regions. Magnetic tension stabilizes strong fields against the MRI, so the corona is laminar and the disk is turbulent. The contour $2h$ (blue) roughly corresponds to $\beta = 1$. The dashed red line marks the ISCO.

5.2 The boundary between disk and corona

The boundary between disk and corona can be identified with the contour where the gas-to-magnetic pressure ratio $\beta = 1$, where the pressure switches from predominantly gas to magnetic. This is plotted in green in Figure 6. The corona is located approximately $2h$ away from the midplane (blue contour). The flow is turbulent in the disk and laminar in the corona because strong fields are stabilized against the MRI by magnetic tension (Pessah & Psaltis 2005). Working in the Boussinesq approximation and ignoring magnetic curvature terms, Balbus & Hawley (1991, 1998) argued the MRI cannot operate when $\beta < 1$.

The plunging region inside the ISCO is similar to the corona, although it is highly magnetized for a different reason. The corona is highly magnetized because magnetic buoyancy raises fields out of the disk. The region inside the ISCO is highly magnetized because plunging fluid stretches frozen-in field lines.

5.3 Effective α

Viscosity in the simulation is generated by MRI-driven turbulence. We define the effective α :

$$\alpha \equiv \frac{\langle W \rangle}{2 \langle p \rangle h r}. \quad (5.5)$$

The height integrated stress, $\langle W \rangle$, is computed from the GRMHD stress-energy tensor (Penna et al. 2010):

$$\langle W \rangle = \frac{1}{2\pi\Delta t} \int_t^{t+\Delta t} \int_{\pi/2+2h}^{\pi/2-2h} \int_0^{2\pi} T_{\hat{r}\hat{\phi}}^{\text{GRMHD}} \sqrt{-g} dt d\theta d\phi. \quad (5.6)$$

The pressure, $\langle p \rangle$, is a ρ^2 -weighted height average:

$$\langle p \rangle = \frac{\int p \rho^2 \sqrt{-g} dt d\theta d\phi}{\int \rho^2 \sqrt{-g} dt d\theta d\phi}. \quad (5.7)$$

The effective α is plotted as a function of radius in Figure 7. We use $\alpha = 0.3$, the effective α at the ISCO, to compare thin disk solutions with the GRMHD disk.

5.4 Comparison

Figure 8 compares the GRMHD disk to thin disk solutions with $M = 10M_\odot$. We assume $\dot{M}/\dot{M}_{\text{edd}} = 0.5$, which Kulkarni et al. (2011) estimated to be the effective accretion rate of this simulation. The GRMHD profiles are only shown out to $r = 9M$ because the

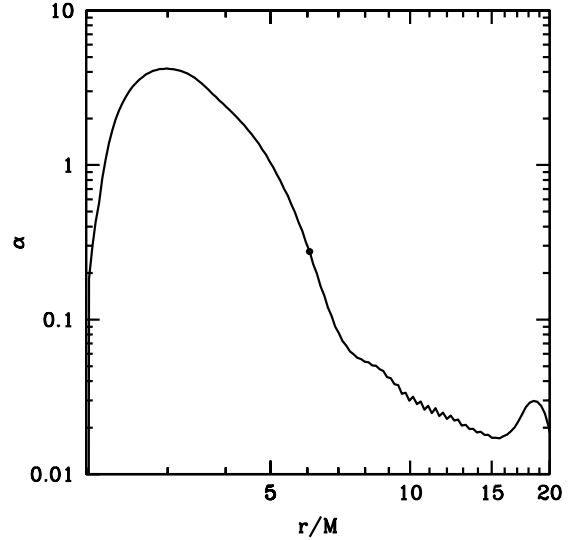


Figure 7. Effective α of the fiducial GRMHD disk simulation. The ISCO is marked with a black dot. There is a sharp rise inside the ISCO, where the plunging fluid stretches the field lines. Unlike slim disk and thin disk models, the GRMHD disk does not have a constant α .

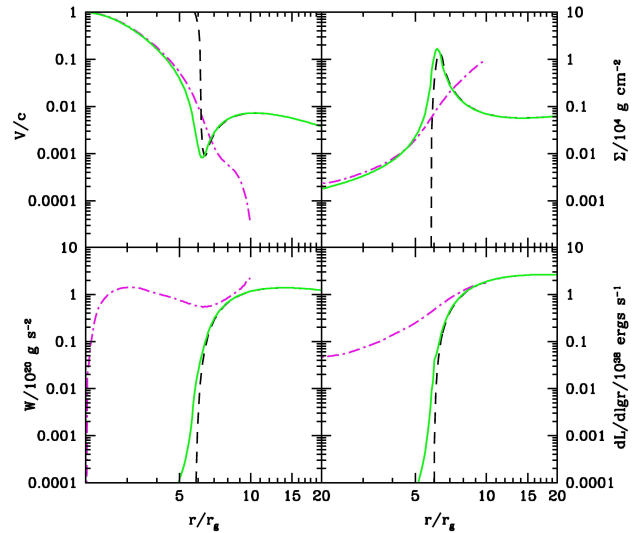


Figure 8. Radial velocity, surface density, stress, and luminosity versus radius for the fiducial GRMHD disk simulation (dot-dashed red) and NT (dashed black) and generalized thin disk models (solid green) with $\alpha = 0.3$ and $\dot{M}/\dot{M}_{\text{edd}} = 0.5$. The GRMHD curves are truncated at $r = 9M$, beyond which the simulations have not reached steady state. (Kulkarni et al. 2011).

simulations are not converged beyond this. Simulation data is time-averaged over the steady state period from $t = 21000M - 26300M$ (Kulkarni et al. 2011; Penna et al. 2010).

The radial velocity and surface density are well-described in the plunging region. This validates the thin disk approximations made in §2. In particular, the GRMHD plunging region is well approximated by hydrostatic equilibrium, with inner edge and sonic point at the ISCO, and geodesic motion.

Outside the ISCO the stress is primarily turbulent in origin. Inside the ISCO, it is generated by mean magnetic fields, which are stretched and amplified by the plunging fluid. Magnetic field reconnection at the grid scale and shocks (Beskin & Tchekhovskoy 2005) create luminosity inside the ISCO. The analytical disk models do not contain magnetic fields, so this physics is not captured. This is why the GRMHD stress and luminosity are not well described by the analytical models inside the ISCO. GRMHD disks are thicker inside the ISCO for the same reasons.

6 CONCLUSIONS

We have developed an analytical model for thin disk accretion in the Kerr metric which generalizes the NT model in three ways: (i) it incorporates nonzero stresses at the inner edge of the disk, (ii) it extends into the plunging region, and (iii) it uses the correct vertical gravity formula. The free parameters are unchanged. Our model is a special case of slim disk solutions, in the regime $h \ll \alpha$ and $\dot{M}_{\text{edd}}/\dot{M} < 0.3$. Under these conditions, energy advection is less important than the stress at the inner edge of the disk, and the inner edge, sonic point, and ISCO are at approximately the same position.

The boundary condition is supplied by setting the radial velocity at the ISCO equal to the sound speed. In the limit $h \rightarrow 0$, this reduces to the NT zero-stress boundary condition. Outside the ISCO, the stress and radiant flux are the sum of the NT prediction and a correction term which incorporates the stress at the ISCO. Inside the ISCO, fluid plunges into the black hole and the motion is approximately geodesic. This enables us to estimate the pressure, and then the stress and radiant flux, of the plunging gas. Throughout we assume the fluid is in vertical hydrostatic equilibrium and the stress is described by an α -viscosity. The model compares favorably with slim disk solutions.

We fit our disk solutions to a GRMHD disk simulation. We argued that the $\beta = 1$ contour is a natural boundary between the disk and coronal regions in GRMHD simulations. Fluid in the disk is turbulent. Outside the disk, where the field is strong and the MRI not operate, the flow is laminar.

The velocity and surface density are well-modeled inside the ISCO. This validates our assumptions that the fluid is in hydrostatic equilibrium and the velocities are nearly geodesic.

The GRMHD plunging region stress is larger than the stress in hydrodynamic models. The stresses are carried by large scale, mean magnetic fields. Some of this stress is dissipated by magnetic reconnection at the grid scale. The slim and thin disk models do not include magnetic fields, so they cannot model this.

Black hole spin parameters can be measured by modeling X-ray spectra using the NT accretion disk (Zhang et al. 1997; Shafee et al. 2006; Davis et al. 2006; Gou et al. 2009, 2010; Steiner et al. 2009). The NT model assumes advection is negligible, the disk inner edge is at the ISCO, and there are no torques at the ISCO. Slim disk solutions are available which do not make these assumptions. Straub et al. (2011) replaced the NT disk with a slim disk model and revisited the spin estimate of LMC X-3. They were unable to improve the estimate because theoretical improvements in the slim disk model were smaller than observational errors.

Observational errors in black hole spin measurements come from uncertainty in the black hole mass, distance, and disk inclination (McClintock et al. 2006). Current observational uncertainties in spin estimates are at best $\Delta a_* \pm 0.2$ at low spins and $\Delta a_* \pm 0.05$ at high spins (Gou et al. 2009). These estimates are made using

data with $\dot{M}/\dot{M}_{\text{edd}} < 0.3$. Kulkarni et al. (2011) created mock data from a GRMHD simulation and fitted it with a NT disk, and computed an estimate of the spin error coming from disk theory. At $\dot{M}/\dot{M}_{\text{edd}} \sim 0.5$ and $\alpha \sim 0.3$, they found spin errors of $\Delta a_* \pm 0.2$ at low spins and $\Delta a_* \pm 0.01$ at high spins. Errors increase with luminosity, so the theoretical uncertainties are always smaller than the observational ones.

This means the NT disk is sufficient for spin measurements at present. However, more sophisticated disk models will be needed as black hole mass, distance, and disk inclination measurements improve. In the hierarchy of disk models, the model in this paper contains more physics than NT but less than a slim disk. Our model and the slim disk are similar when $\dot{M}/\dot{M}_{\text{edd}} > 0.3$ and $h < \alpha$, but our model is analytical, so it might be simpler to use in some cases.

McClintock et al. (2006) introduced the selection criterion $\dot{M}/\dot{M}_{\text{edd}} < 0.3$ when they measured the spin of the black hole GRS 1915+105. Black hole X-ray binaries have variable luminosities and the NT model is only valid at low luminosities. Of the 22 observations of GRS 1915+105 available to McClintock et al. (2006), five satisfied the $\dot{M}/\dot{M}_{\text{edd}} < 0.3$ criterion. These five observations gave a nearly consistent spin parameter $a_* > 0.98$. Observations with $\dot{M}/\dot{M}_{\text{edd}} > 0.3$ give inconsistent spin results, but the NT model is not valid in this regime.

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APPENDIX A: THE KERR METRIC

We assume spinning black holes can be described by the Kerr metric and the accretion disk lies in the equatorial plane of the metric. The Kerr metric in Boyer-Lindquist coordinates (t, r, θ, ϕ) in and near the equatorial plane ($|\theta - \pi/2| \ll 1$) is:

$$\begin{aligned} ds^2 &= -\frac{r^2 \Delta}{A} dt^2 + \frac{A}{r^2} (d\phi - \omega dt)^2 + \frac{r^2}{\Delta} dr^2 + dz^2, \\ \Delta &= r^2 - 2Mr + a^2, \\ A &= r^4 + r^2 a^2 + 2Mra^2, \\ \omega &= 2Mar/A. \end{aligned} \quad (\text{A1})$$

Here M and a are the mass and specific angular momentum of the hole. We have replaced the usual angular coordinate by $z = r \cos \theta \approx r(\theta - \pi/2)$. Define the auxiliary parameters:

$$a_* = a/M \quad (\text{note: } -1 \leq a_* \leq +1), \quad (\text{A2a})$$

$$x_1 = 2 \cos(\cos^{-1}(a_*)/3 - \pi/3), \quad (\text{A2b})$$

$$x_2 = 2 \cos(\cos^{-1}(a_*)/3 + \pi/3), \quad (\text{A2c})$$

$$x_3 = -2 \cos(\cos^{-1}(a_*)/3), \quad (\text{A2d})$$

and a dimensionless radial coordinate

$$x = (r/M)^{1/2}. \quad (\text{A3})$$

For simplicity in splitting formulae into Newtonian limits plus relativistic corrections, we shall introduce the following functions of x and a_* with value unity far from the hole:

$$\mathcal{A} = 1 + a_*^2 x^{-4} + 2a_*^2 x^{-6}, \quad (\text{A4a})$$

$$\mathcal{B} = 1 + a_* x^{-3}, \quad (\text{A4b})$$

$$\mathcal{C} = 1 - 3x^{-2} + 2a_*^2 x^{-3}, \quad (\text{A4c})$$

$$\mathcal{D} = 1 - 2x^{-2} + a_*^2 x^{-4}, \quad (\text{A4d})$$

$$\mathcal{E} = 1 + 4a_*^2 x^{-4} - 4a_*^2 x^{-6} + 3a_*^4 x^{-8}, \quad (\text{A4e})$$

$$\mathcal{F} = 1 - 2a_* x^{-3} + a_*^2 x^{-4}, \quad (\text{A4f})$$

$$\mathcal{G} = 1 - 2x^{-2} + a_* x^{-3}, \quad (\text{A4g})$$

$$\mathcal{H} = 1 - 2x^{-2} + 2a_* x^{-2} x_0^{-1} \mathcal{F}_0 \mathcal{G}_0, \quad (\text{A4h})$$

$$\mathcal{J} = \mathcal{A} - 2a_* x^{-6} x_0 \mathcal{F}_0 \mathcal{G}_0^{-1}, \quad (\text{A4i})$$

$$\mathcal{J} = \mathcal{C} - x^{-2} \mathcal{J}^{-1} \left[1 - a_* x_0^{-1} \mathcal{F}_0^{-1} \mathcal{G}_0 + a_*^2 x^{-2} \mathcal{J} \mathcal{J}^{-1} \right. \\ \left. \times \left(1 + 3x^{-2} - 3a_*^{-1} x^{-2} x_0 \mathcal{F}_0 \mathcal{G}_0^{-1} \right) \right], \quad (\text{A4j})$$

$$\mathcal{K} = \left| \mathcal{A} \mathcal{J} \left(1 - x^{-4} \mathcal{A}^2 \mathcal{D}^{-1} \left(x_0 \mathcal{F}_0 \mathcal{G}_0^{-1} \mathcal{C} - 2a_* x^{-2} \mathcal{A}^{-1} \right)^2 \right)^{-1} \right|, \quad (\text{A4k})$$

$$\mathcal{O} = \mathcal{H} \mathcal{J}^{-1}, \quad (\text{A4l})$$

$$\mathcal{Q} = \mathcal{B} \mathcal{C}^{-1/2} \frac{1}{x} \left[x - x_0 - \frac{3}{2} a_* \ln \left(\frac{x}{x_0} \right) \right. \\ \left. - \frac{3(x_1 - a_*)^2}{x_1(x_1 - x_2)(x_1 - x_3)} \ln \left(\frac{x - x_1}{x_0 - x_1} \right) \right. \\ \left. - \frac{3(x_2 - a_*)^2}{x_2(x_2 - x_1)(x_2 - x_3)} \ln \left(\frac{x - x_2}{x_0 - x_2} \right) \right. \\ \left. - \frac{3(x_3 - a_*)^2}{x_3(x_3 - x_1)(x_3 - x_2)} \ln \left(\frac{x - x_3}{x_0 - x_3} \right) \right], \quad (\text{A4m})$$

$$\mathcal{R} = \mathcal{F}^2 \mathcal{C}^{-1} - a_*^2 x^{-2} (\mathcal{G} \mathcal{C}^{-1/2} - 1), \quad (\text{A4n})$$

$$\mathcal{S} = \mathcal{A}^2 \mathcal{B}^{-2} \mathcal{C} \mathcal{D}^{-1} \mathcal{R}, \quad (\text{A4o})$$

$$\mathcal{V} = \mathcal{D}^{-1} \left[1 + x^{-4} (a_*^2 - x_0^2 \mathcal{F}_0^2 \mathcal{G}_0^{-2}) + 2x^{-6} (a_* - x_0 \mathcal{F}_0 \mathcal{G}_0^{-1}) \right]. \quad (\text{A4p})$$

A subscript 0 indicates the quantity is evaluated at the innermost stable circular orbit (ISCO). Functions \mathcal{A} - \mathcal{G} and \mathcal{Q} are taken from Novikov & Thorne (1973) and Page & Thorne (1974).

A1 Geodesics

The non-zero components of the four-velocity, u^μ , for general equatorial, timelike geodesic motion in the Kerr metric are (Chandrasekhar 1992):

$$u^t = \frac{1}{\Delta} \left[\left(r^2 + a^2 + \frac{2a^2 M}{r} \right) E - \frac{2aM}{r} L \right], \quad (\text{A5a})$$

$$u^r = \frac{1}{r^2} \left[r^2 E^2 + \frac{2M}{r} (aE - L)^2 + (a^2 E^2 - L^2) - \Delta \right], \quad (\text{A5b})$$

$$u^\phi = \frac{1}{\Delta} \left[\left(1 - \frac{2M}{r} \right) L + \frac{2aM}{r} E \right], \quad (\text{A5c})$$

where E and L are the conserved specific energy and angular momentum of the motion. Circular geodesics have energy per unit mass

$$E = |u_t| = \mathcal{G} / \mathcal{C}^{1/2}, \quad (\text{A6})$$

angular momentum per unit mass

$$L = u_\phi = M^{1/2} r^{1/2} \mathcal{F} / \mathcal{C}^{1/2}, \quad (\text{A7})$$

and angular velocity

$$\Omega = \frac{u^\phi}{u^t} = \frac{M^{1/2}}{r^{3/2} + aM^{1/2}} = \frac{M^{1/2}}{r^{3/2}} \frac{1}{\mathcal{B}}. \quad (\text{A8})$$

Circular geodesics are unstable inside the ISCO. The radius of the ISCO is:

$$r_0/M = 3 + Z_2 - [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}, \quad (\text{A9})$$

$$Z_1 = 1 + \left(1 - a_*^2 \right)^{1/3} \left[(1 + a_*)^{1/3} + (1 - a_*)^{1/3} \right],$$

$$Z_2 = \left(3a_*^2 + Z_1^2 \right)^{1/2}$$

The linear velocity of a circular orbit relative to a locally nonrotating observer (Bardeen et al. 1972) is

$$V_{(\phi)} = \frac{A}{r^2 \Delta^{1/2}} (\Omega - \omega). \quad (\text{A10})$$

The Lorentz factor corresponding to this linear velocity is

$$\gamma = \left(1 - V_{(\phi)}^2 \right)^{-1/2}. \quad (\text{A11})$$

The only nonzero components of the fluid frame shear tensor for the congruence of circular, equatorial geodesics are

$$\sigma_{\hat{r}\hat{\phi}} = \sigma_{\hat{\phi}\hat{r}} = \frac{1}{2} \frac{A}{r^3} \gamma^2 \Omega_{,r}. \quad (\text{A12})$$

APPENDIX B: DISK STRUCTURE EQUATIONS

B1 Definitions

The stress-energy tensor of a relativistic fluid is

$$\mathbf{T} = \rho(1 + \Pi) \mathbf{u} \otimes \mathbf{u} + \mathbf{t} + \mathbf{u} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{u}, \quad (\text{B1})$$

where ρ is rest mass density in the local rest frame of the baryons (LRF), Π is the specific internal energy in the LRF, \mathbf{t} is the stress tensor in the LRF, and \mathbf{q} is the energy flux relative to the LRF. We make the thin disk approximation $\Pi = 0$ (c.f. §2.1).

The disk structure equations are expressed in terms of the surface density of the disk,

$$\Sigma = \int_{-H}^{+H} \rho dz = 2\rho H, \quad (\text{B2})$$

the integrated shear stress,

$$W = \int_{-H}^{+H} t_{\hat{\phi}\hat{r}} dz = 2t_{\hat{\phi}\hat{r}} H, \quad (\text{B3})$$

the radial velocity of the gas in the locally non-rotating frame,

$$v^{\hat{r}} = \sqrt{g_{rr}} u^r, \quad (\text{B4})$$

and the the flux of radiant energy off the upper face of the disk,

$$F = q^{\hat{z}}(z = +H) = q^{\hat{z}}(z = -H), \quad (\text{B5})$$

where the disk scale height, H , is defined by $h = H/r$.

B2 Radial structure equations

The radial structure of the disk is controlled by conservation of baryon number, conservation of angular momentum, and conservation of energy:

$$(\rho u^\mu)_{;\mu} = 0, \quad (\text{B6})$$

$$(T^\mu_\phi)_{;\mu} = 0, \quad (\text{B7})$$

$$(T^\mu_t)_{;\mu} = 0. \quad (\text{B8})$$

Integrating (B6) gives the accretion rate of a stationary disk:

$$\dot{M} = -2\pi r \Sigma u^r = (\text{constant independent of } r \text{ and } t). \quad (\text{B9})$$

Combining angular momentum and energy conservation gives (c.f. §2.1):

$$F = -\sigma_{\hat{r}\hat{\phi}} W. \quad (\text{B10})$$

Combining all three conservation laws gives an integral solution for the flux:

$$4\pi r \frac{(E - \Omega L)^2}{-\Omega_r} F / \dot{M} = \int_{r_0}^r (E - \Omega L) L_r dr + C. \quad (\text{B11})$$

Page & Thorne (1974) give an analytical formula for the integral on the RHS when $r \geq r_0$. The integration constant C is related to the flux at the ISCO. The NT no-torque boundary condition is $C = 0$. We allow nonzero C .

B3 Vertical structure equations

The vertical structure of the disk is controlled by pressure balance (c.f. §2.3),

$$-\frac{p}{\rho} + h^2 \frac{\mathcal{L}_z^2}{r^2} = 0, \quad (\text{B12})$$

$$\mathcal{L}_z^2 = u_\phi^2 - a^2 (u_t - 1),$$

the Shakura-Sunyaev α -viscosity prescription,

$$t_{\hat{\phi}\hat{r}} = \alpha p \quad (\text{B13})$$

radiative energy transport,

$$bT^4 = \bar{\kappa} \Sigma F, \quad (\text{B14})$$

the equation of state,

$$p = p^{(\text{rad})} + p^{(\text{gas})}, \quad (\text{B15})$$

$$p^{(\text{rad})} = \frac{1}{3} b T^4,$$

$$p^{(\text{gas})} = \rho (T/m_p),$$

and the opacity law,

$$\bar{\kappa} = \bar{\kappa}_{ff} + \bar{\kappa}_{es}, \quad (\text{B16})$$

$$\bar{\kappa}_{ff} = (0.64 \times 10^{23}) \left(\frac{\rho}{\text{g/cm}^3} \right) \left(\frac{T}{K} \right)^{-7/2} \frac{\text{cm}^2}{g},$$

$$\bar{\kappa}_{es} = 0.40 \frac{\text{cm}^2}{g}.$$

B4 Solving for the disk structure

At this point the disk structure is defined by seven equations (B9)-(B16) for nine unknowns $E, L, u^r, W, F, \Sigma, p, h, T, \bar{\kappa}$ and four free parameters M, a, \dot{M}, α . The integration constant C is fixed by the boundary condition (C8).

To close the problem we need two more relations. These are prescriptions for E and L . Outside the ISCO, the disk nearly follows circular geodesics so E and L are (A6) and (A7). Inside the ISCO, the disk follows non-circular plunge trajectories with constant E and L defined in §2.2.

The fluid flow is slightly non-geodesic because it is acted upon by stresses. These are small deviations because the disk is thin and there are several ways of treating them. The different prescriptions are equivalent in the thin disk limit, so we choose the simplest.

Outside the ISCO, we use the angular velocity of circular geodesics (A8), but do not enforce $u^r = 0$ (as would be required if the flow were truly geodesic by (A5b)). We use this angular velocity when computing the shear tensor (A12).

Inside the ISCO, we use the geodesic velocities (A5a)-(A5c), but do not assume the radiant flux integral (B11) is zero (as would be required if L were truly constant). This eliminates one independent variable from the problem in the plunging region (because u^r is fixed by the geodesic equations) and one of the disk structure equations (because we do not enforce (B11)).

Throughout most of the plunging region, the angular velocity exceeds the radial velocity, so we use (A5c) and (A12) to compute the shear. This fails near the photon orbit, but the thin disk approximations are expected to break down there (§2.1).

Explicit solutions are in §3.

APPENDIX C: SCALING OF COMPRESSION, ADVECTION, AND BOUNDARY STRESS TERMS WITH α AND h

The law of energy conservation (B8) can be rewritten (Novikov & Thorne 1973):

$$\rho \frac{d\Pi}{d\tau} + \nabla \cdot \mathbf{q} = -\sigma_{\alpha\beta} t^{\alpha\beta} - \frac{1}{3} \theta t^\alpha_\alpha - \mathbf{a} \cdot \mathbf{q}. \quad (\text{C1})$$

We have introduced the convective derivative $d/d\tau \equiv \mathbf{u} \cdot \nabla$, the scalar expansion $\theta \equiv \nabla \cdot \mathbf{u}$, the acceleration vector $\mathbf{a} \equiv \nabla_\mathbf{u} \mathbf{u}$, the shear tensor

$$\sigma_{\alpha\beta} \equiv \frac{1}{2} (u_{\alpha;\mu} h^\mu_\beta + u_{\beta;\mu} h^\mu_\alpha) - \frac{1}{3} \theta h_{\alpha\beta}, \quad (\text{C2})$$

and the projection tensor

$$h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta. \quad (\text{C3})$$

Each term in (C1) has a simple physical interpretation. The term $\mathbf{a} \cdot \mathbf{q}$ is a special relativistic correction associated with the inertia of the flowing energy \mathbf{q} . We assume \mathbf{q} is directed entirely along z and $u^z = 0$, so $\mathbf{a} \cdot \mathbf{q} = 0$.

The remaining terms on the RHS of (C1) correspond to energy generation by shear stresses, $-\sigma_{\alpha\beta} t^{\alpha\beta}$, and by compression, $-1/3 \theta t^\alpha_\alpha$. The sink terms on the LHS of (C1) describe energy advection, $-\rho d\Pi/d\tau$, and radiative losses, $\nabla \cdot \mathbf{q}$.

After height integrating and normalizing by \dot{M} , the compression term scales as

$$\frac{\theta t^\alpha_\alpha h}{\dot{M}} \propto \frac{u^r_r p h}{\rho u^r h} \propto h^2. \quad (\text{C4})$$

Here \propto means proportionality with respect to h and α , which are considered small. So, for example, $u^r_r/u^r \propto (u^r/r)/u^r \propto 1$. In the first step of (C4), we inserted the accretion rate (B9). In the second step, we used the pressure balance relation (B12).

The height integrated advection term scales as

$$\frac{\rho d\Pi/d\tau h}{\dot{M}} \propto \frac{\rho u^r \Pi_r h}{\rho u^r h} \propto h^2, \quad (\text{C5})$$

where we have used $\Pi \propto p/\rho \propto h^2$.

The height-integrated stress at the ISCO enters the solution as a boundary condition when integrating the energy equation (C1). It scales as

$$\frac{\sigma_{\hat{r}\hat{\phi}} \hat{r}^{\hat{\phi}} h}{\dot{M}} \propto \frac{\alpha p h}{\rho u' h} \propto \alpha h. \quad (\text{C6})$$

In the first step, we used the α -viscosity prescription (B13). In the second step, we identified the ISCO with the sonic point of the disk (c.f. §2.1), so $u' \propto c_s \propto \sqrt{p/\rho} \propto h$ (by B12).

Equations (C4)-(C6) give the scaling of compression, advection, and boundary stresses with h and α . In the NT limit, $h \rightarrow 0$, all three terms vanish. Under the weaker assumption $h \ll \alpha$, compression and advection are small but the stress at the ISCO is important. So we obtain a self-consistent generalization of the NT model by ignoring advection and compression but including the boundary stress at the ISCO, when $h \ll \alpha$.

Dropping advection and compression terms from the energy equation (C1), we have

$$\frac{dq^z}{dz} = -\sigma_{\hat{r}\hat{\phi}} \hat{r}^{\hat{\phi}}, \quad (\text{C7})$$

which says energy generated by shear stresses is immediately radiated away. Height integrating gives (B10).

Rewriting the stress at the ISCO (C6) as a radiant flux using (B10) fixes the boundary term in the disk solution (B11):

$$C = \left[\alpha h \gamma (E - \Omega L) \frac{\mathcal{L}_*}{r} \right]_0. \quad (\text{C8})$$

This reduces to the NT choice $C = 0$ in the razor thin limit $h \rightarrow 0$. However in general the flux and stress at the ISCO will not be zero.

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